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Evolution of Predicate Logic Notation

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Abstract. While there is a lot of research on the ideas behind different systems of predicate logic, there is no research concerning the history of its notation. In my paper, I analyze the history of the predicate logic notation and establish the main characteristics of the processes the notation has undergone in the past two centuries. I present the development of predicate logic within the two traditions of notations, Fregean and Boolean, and their subsequent unification into one. I argue that the term "evolution" can be applied to the processes within the development of predicate logic notation. I then analyze the evolution of predicate logic notation to determine which term can characterize the process more precisely: "cultural evolution" or "language evolution".

Keywords: Predicate logic · Language evolution · Cultural evolution

1 Why study predicate logic evolution?

Predicate logic has a manifold history of emergence and change. Put on stage in the middle of the nineteenth century, formal predicate logic developed simultaneously within two traditions: Boolean and Fregean, both of which noticeably differs from the type of logic we use now. In the present paper, I analyze the evolution of the predicate logic (further - PL) notation and establish the main characteristics of the process PL notation has undergone in the past two centuries.

A reasonable question could be raised: why do we need to look at the evolution of predicate logic? The answer is the same as "Why do we need to look at the evolution of natural language?": since we are interested in how a specific structure (either natural language or predicate logic) works, we need to look at the processes engaged in the emergence of the structure. A follow-up question arises here: why do we need to look at the *structure* of predicate logic? An answer to this one is connected, again, to the natural language, at least within this paper: at the moment, there are various attempts to analyze, model, and study natural language using different modifications of predicate logic [15],[7],[8],[2]. At the moment, the predicate calculus (with respect to its different modifications) is not considered to be fully adequate for the modeling of natural language [2]. Some researchers[1] argue that there may be an enrichment strong enough for

[1] Ben-Yami (2019), pp. 5-8.

that purpose. To establish whether there could be an enrichment possessing the specified characteristic, one must have an in-depth knowledge of the PL system and its capabilities. I argue that the study of PL evolution deepens our knowledge of the structure of the logic just as the study of natural language evolution deepens our knowledge of the structure of natural language.

In this paper, I only analyze a part of the PL history: the evolution of its formal notation. I do that for two reasons. First, while I was able to find works on how ideas behind PL systems have changed through XIX-XX centuries [11],[3],[4]. I was not able to find surveys including the history of PL notation. Second, the analysis of PL evolution as a whole would take a lot more space than one paper. However, I agree that the history of PL notation is inseparable from the history of ideas behind every specific system of notation. Hence, in this paper, I present not a completed study, but the first step of the research project.

There is no consensus on whether language evolution equals cultural evolution since some researchers consider language evolution to be a more complicated phenomenon [18]. Nevertheless, language evolution stands out from some typical features of cultural evolution: for instance, natural language has no preset direction of evolving, unlike almost any other cultural unit. Considering the way the research question is formulated, it is important to remember that there is no *clear* distinction between cultural and language evolution.

2 Boolean paradigm

There are two traditions in the history of predicate logic: the one starting with George Boole and the one starting with Gottlob Frege.

Two milestones of the history of predicate logic were the introductions of (a) many-place predicates and (b) quantifiers: both universal and existential[2]. In the Boolean tradition, (a) is considered to be done by Augustus De Morgan (1864) and (b) was presented by different authors, including C. S. Peirce (1885), G. Peano (1889), and D. Hilbert (1917/1918). G. Frege introduced both (a) and (b) independently in *Begriffsschrift*, which I will discuss in the next section. In this section, I am going to talk about the Boolean branch.

In 1847, Boole published *The Mathematical Analysis of Logic*, where he attempted to translate Aristotelian syllogistic logic into an algebraic calculus consisting of classes and propositions [5]; it is, basically, what we call Boolean logic today. Boole could express some rudimental connectives in his system: for instance, Aristotelian "All X are Y" in Boolean calculus would be denoted as "$xy = x$", where multiplication "xy" represents logical conjunction "$x \wedge y$".

Peirce continued to work within the Boolean tradition, which he enriched by adding and introducing some notations Boole did not have. Firstly, Peirce used $\therefore$ as a symbol of illation that separates the premises of a syllogism from its conclusion (Figure 1).

[2] Ewald & Sieg (2013), p. 31-32.

$$\begin{array}{c} \text{P} \\ \therefore \text{C} \end{array}$$

Fig. 1: The passage from the premise(s) P to the conclusion C.

Secondly, in *On the Algebra of Logic*, Peirce used the symbol $a \prec b$, firstly introducing it as an implication: "The symbol $P_i \prec C_i$ is the copula, and signifies primarily that every state of things in which a proposition of the class P_i is true is a state of things in which the corresponding propositions of the class C_i are true" (Peirce (1885), p.18). Later in the paper, Peirce defines universal quantification using only $\prec$ symbol (Table 1).

Table 1: Quantifiers notation in Peirce (1885).

$a \prec b$	Every a is b	Universal affirmative
$a \prec \bar{b}$	No a is b	Universal negative
$\breve{a} \prec b$	Some a is b	Particular affirmative
$\breve{a} \prec \bar{b}$	Some a is not b	Particular negative

Therefore, we can see that Peirce does not have a unique notation for either an implication or the universal quantifier. In some cases Peirce used Π to denote the universal quantifier specifically, but $\prec$ is more common within complex statements. The author uses $\prec$ as a general sign of inference and corollary. The situation is similar for the existential quantifier, which Peirce denotes by a semicircle over an antecedent of $\prec$. The logician also used Σ, which was originally introduced to denote a logical sum, to represent *some* of a class[3] – that notation could be considered as an introduction of the existential quantifier. Nevertheless, just as in the case of the universal quantifier, Peirce did not establish one single notation to denote the logical quantifier *Some*.

In the same work, *On the Algebra of Logic*, Peirce defines the existential quantifier through negation of the copula: "[...] we may write $\breve{S} \prec P$ for $S \bar{\prec} \bar{P}$"(Peirce (1885), p. 29). Therefore, we can state that Peirce, while working with Boolean calculus, introduced a notation for the existential quantifier, but did not differ between an implication and the universal quantifier, as well as he did not have a unique notation for any of the two. One can draw a parallel here with the Fregean notation since in *Begriffsschrift* Frege only introduces the universal quantifier and defines the existential quantifier by the negation of the universal one. Both Frege and Peirce based their work on Aristotelian syllogistic logic, and that makes it even more interesting that the authors decided to take these independent yet parallel paths.

[3] Beatty (1969), p. 67.

3 Fregean paradigm

In 1879, Frege introduced many-places predicates and the universal quantifier in *Begriffsschrift*. The author was working, as was said, independently from Boole and Pierce, who were trying to build a calculus based on Aristotelian logic. Frege's intentions were similar: he wanted to build a formal language "of pure thought" based on Aristotelian syllogistic logic. However, the structure and notation of *Begriffsschrift* are a lot different from the structure and notation of Boole's *Mathematical Analysis of Logic* and Peirce's *On the Algebra of Logic*, and the present section will discuss that difference.

In the first chapter of *Begriffsschrift*, Frege introduces a special sign to represent a judgment:

$$\vdash\!\!\!-\!\!\!-$$

Let $\vdash\!\!\!-\!\!\!- A$ stand for the judgment "Opposite magnetic poles attract each other", says Frege. Then

$$-\!\!\!-\!\!\!- A$$

will express "merely the idea of the mutual attraction of opposite magnetic poles" (Frege (1879), p.11).

Unlike Peirce, who used $\prec$ or $\therefore$ to represent the existence of a judgment – symbols, which original purpose is representing a formal inference, – Frege introduced $\vdash\!\!\!-\!\!\!-$ exclusively as a special sign for judgment representation from the very beginning. Every element of a specific judgment is always embedded in $\vdash\!\!\!-\!\!\!-$.

Frege also introduced a notation for the universal quantifier:

$$\vdash\!\!\overset{\mathfrak{a}}{\smile}\!\!-$$

where a is a variable and a small pit on a horizontal line is a symbol of a quantifier. The existential quantifier in *Begriffsschrift* was expressible through the universal one and a negation (Figure 3).

As in Peirce (1885), the main unit of an asserted sentence in Frege (1879) was an implication, with only the exception that Frege strictly distinguished between implication as a connective and a quantifier as a unit specifying the domain of discourse. An implication in Frege was presented the following way (Figure 2).

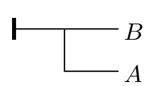

Fig. 2: If A, then B[4].

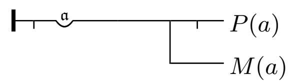

Fig. 3: Fregean notation for $\neg\forall(a)(M(a) \to \neg P(a))$ or $\exists(a)(M(a) \land P(a))$.

As we can see so far, Boolean and Fregean traditions, while having very different formal notations, came to the two similar grounds: (a) the main binary connective in the sentence structure is an implication; (b) both traditions, nevertheless there are two domain-specifying words in Aristotelian logic (*All* and *Some*), have a unique notation for only one quantifier (either universal or existential) and denote another quantifier with a negation of the available one. However, the important distinction is that Frege's notation was based on the notions of a function and a variable, both of which were able to be quantified, while Peirce's notation did not go that far and was only able to quantify variables[5].

4 The integration of two traditions

While Frege's *Begriffsschrift* was not yet recognized by neither philosophers nor mathematicians, the Boolean tradition was growing further. In *Arithmetices principia* (1889), Giuseppe Peano introduced, among many other symbols, $\exists$ – the notation for the existential quantifier logicians still use these days. $\exists$ was later adopted by Whitehead and Russell in *Principia Mathematica* [**?**], the authors of which, while being familiar with Frege's notation, preferred to pursue their work using Boolean-style notation. Since Russell worked in type theory, he found Frege's formal distinctions to first and higher-order logics useful in his work. However, *Principia Mathematica* did not inherit anything from the formal notation of *Begriffsschrift* other than $\vdash\!\!\!\longrightarrow$, which was adapted into a small version $\vdash$ and used for the representation of an asserted proposition.

Principia Mathematica was the first work dedicated (up to some point) to the predicate logic that contained both a unique symbol for the existential quantifier and a unique symbol for the universal quantifier. The universal quantifier was introduced by Whitehead in a form of two brackets around a variable: (a). Only later, in his course *Prinzipien der Mathematik* (1917-1918), David Hilbert combined predicate logic into something the modern logicians would find familiar. Hilbert used originally Boolean notation which Russell and Whitehead combined with Fregean logical hierarchy in *Principia Mathematica*, adapted it for the predicate logic specifically, and added some new notations, including $\forall$ notation for the universal quantifier. We can see that predicate logic notation, which originated from the Aristotelian syllogistic logic and took two parallel independent paths at the end of the 19th century, came back to a unified system, which inherited its theoretical grounds from the Fregean tradition, and its

[5] Ewald & Sieg (2013), p. 40.

basic notation from the Boolean tradition. Almost in the same form as Hilbert presented it in *Prinzipien der Mathematik*, the notation has survived to this day.

5 Characteristics of PL notation evolution

We can check whether PL notation evolution is Darwinian. There are three preconditions (variation, competition, inheritance) and three additional characteristics (adaptation, maladaptation, convergence) in Darwinian evolution[6]. PL notation evolution is variative since there are at least two different independently developing branches taking place: Fregean and Boolean traditions of notation. PL notation evolution is competitive because it can be clearly seen how Boolean and Fregean notations compete against each other within different philosophical and mathematical schools. Moreover, we can see how Boolean tradition 'wins' due to certain practical and theoretical circumstances. The notation evolution also has the characteristic of inheritance, since we can see, at least in the Boolean tradition, that Boole's algebraic calculus notation was inherited in Peirce (1885), Peano (1889), De Morgan (1864), and further.

We see adaptation processes in PL notation evolution: the universal quantifier, firstly introduced by Peirce (1885), was not different from Peirce's implication sign $a \prec b$, which was not convenient for predicate calculus. In 1910, Whitehead introduced the (a) notation for the universal quantifier: it was different from the implication sign, but still not yet convenient enough, since it was easy to confuse (a) as a quantifier and (a) as a variable of a certain predicate. Lastly, in 1917-1918, Hilbert introduced $\forall$, which is still convenient for logicians to use since it cannot be confused with anything else. I can also present an example of maladaptation, which is a presence of a certain feature that is not very convenient and, at the same time, not inconvenient enough for a system to reject it. In predicate logic, this feature is writing $\exists x.M(x)$ instead of $\exists x.Mx$: there is no need now to put additional brackets to the predicate's variable in a formula since it is hard to confuse the variable with anything else, even while working with a many-place predicate. However, that is not a big inconvenience and people still proceed to use rules of predicate logic with a demand for brackets around a predicate's variable. Finally, there are convergent processes in PL notation evolution, such as the emergence of a unique notation for only one quantifier and a negated notation for another one in both Boolean and Fregean traditions. We established that PL notation evolution is Darwinian. Now we can consider whether it is a case of pure cultural evolution or some unique features of language evolution are presented.

First, predicate logic emerged with a different need than natural language, which emergence was demanded by the need for communication, socialization, survival, etc. The emergence of PL arose from an *academic* need. Secondly, PL

[6] Mesoudi (2011), pp. 12-14.

notation had a specific direction of evolving, and, starting with an algebraic calculus, ended up being a complex many-order structure. Language evolution is different: it does not have an *intentionally* specified aim as, for instance, "describe all there is in mathematics with formal logic", therefore language evolves by being used for many different purposes and develops various evolutional branches. However, language evolution has a demand for learnability [12] which ensures the language is preserved within the next generation. We can see a similar pattern in the evolution of predicate logic notation: there is a demand for a predicate calculus to be comprehensible since the logician introducing the notation expects the logicians of the next academic generation not to only preserve their system, but *use it* actively. A demand for comprehensibility or for learnability is not a common feature of cultural evolution: for instance, there is no demand for comprehensibility in art or tradition. PL notation evolution is also cumulative, as almost any cultural evolution, which means that the logicians who inherit predicate calculus systems from the previous generation of logicians can build a new calculus *on top* of the existing one, while in natural language this dynamic sometimes may be observed, but not in an intentional way.

In conclusion, the evolution of predicate logic notation is a case of a pure cultural evolution (no natural need, specific direction of evolving, cumulative, etc.) with one inclusion of a specific language evolution feature – a demand for comprehensibility. Nevertheless, while comprehensibility and learnability are two different demands, the reason for their existence in predicate logic and language evolutions is a need for the next generation to inherit the system the previous generation used. However, this feature so far seems to be the only characteristic that brings PL evolution closer to language evolution.

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